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Field Equations on $SE(k)$-Manifold X_n

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An Einstein connection which is both a special connection and a (k)-connection is called an $SE(k)$-connection. And a generalized even-dimensional Riemannian manifold X_n with the so-called "$SE(k)$-condition" defined by the $SE(k)$-connection is called the $SE(k)$-manifold. We obtain the necessary and sufficient condition that there is a unique $SE(k)$-connection in X_n. Next, using these results, we define the $SE(k)$-manifold and study the properties of the curvature tensors and the field equations in the $SE(k)$-manifold X_n.

1. INTRODUCTION

In Appendix II to his last book Einstein (1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates for the space-time X_4. Although the geometrical consequences of these postulates were not developed very far by Einstein, Hlavatý (1957) gave its mathematical foundation for the first time, characterizing Einstein's unified field theory as a set of geometrical postulates for X_4. Since then the geometrical consequences of these postulates have been developed very far by a number of mathematicians and physicists; among them Hlavatý's contributions are the most distinguished.

Generalizing X_4 to an n-dimensional generalized Riemannian manifold X_n, Wrede (1958) studied principles A and B of Einstein's unified field theory for the first time. But his solution of Einstein's equations is not surveyable, probably due to the complexity of the higher dimensions.

The first purpose of the present paper is to introduce the new concept of even-dimensional $SE(k)$-manifold X_n imposing the so-called "$SE(K)$-condition" on X_n and to find a unique representation of Einstein's connection

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on the $SE(k)$-manifold X_n in a simple and surveyable tensorial form. The second purpose is to study the properties of the curvature tensors and the field equations in the $SE(k)$-manifold X_n.

2. PRELIMINARIES

This section is a brief collecion of basic concepts, results, and notations needed in subsequent considerations.

Let X_n, where n is even, be a generalized n-dimensional Riemannian space referred to a real coordinate system χ^i, which admits only coordinate transformations $\chi^i \rightarrow \bar{\chi}^i$ for which

$$\operatorname{Det}\left(\frac{\partial \bar{\chi}}{\partial \chi}\right) \neq 0 \tag{2.1}$$

where, here and in the sequel, Latin indices take the values $1, 2, \ldots, n$, and follow the summation convention.

The space X_n is endowed with a general real nonsymmetric tensor g_{ij} which may be split into a symmetric part h_{ij} and a skew-symmetric part k_{ij}:

$$g_{ij} = h_{ij} + k_{ij} \tag{2.2}$$

where we assume that

$$G = \operatorname{Det}(g_{ij}) \neq 0 \tag{2.3a}$$

$$H = \operatorname{Det}(h_{ij}) \neq 0 \tag{2.3b}$$

$$K = \operatorname{Det}(k_{ij}) \neq 0 \tag{2.3c}$$

Remark 2.1. (a) According to (2.3b), there is a unique tensor $h^{ik} = h^{ki}$ defined by

$$h_{ij} h^{ik} = \delta_j^k \tag{2.4}$$

The tensors h_{ij} and h^{ik} will serve for raising and/or lowering indices of tensors in X_n in the usual manner.

(b) According to (2.3a), there is a unique tensor

$$*g^{ik} = \frac{\partial \ln G}{\partial g_{ik}} \tag{2.5}$$

satisfying the condition

$$g_{ij}\, {*g^{ik}} = g_{ji}\, {*g^{ki}} = \delta_j^k \tag{2.6}$$

Similarly, there is a unique tensor $*k^{ik} = *k^{[ik]}$ such that

$$k_{ij}\, *k^{ik} = k_{ji}\, *k^{ki} = \delta_j^k \tag{2.7}$$

The space X_n is assumed to be connected by a general real connection Γ_{ij}^k with the following transformation rule:

$$\bar{\Gamma}_{ab}^c = \frac{\partial \bar{\chi}^c}{\partial \chi^k}\left(\frac{\partial \chi^i}{\partial \bar{\chi}^a}\frac{\partial \chi^j}{\partial \bar{\chi}^b}\Gamma_{ij}^k + \frac{\partial^2 \chi^k}{\partial \bar{\chi}^a\, \partial \bar{\chi}^b}\right) \tag{2.8}$$

It may be also decomposed into a symmetric part Λ_{ij}^k and a skew-symmetric part $S_{ij}{}^k$ called the torsion tensor of Γ_{ij}^k,

$$\Gamma_{ij}^k = \Lambda_{ij}^k + S_{ij}{}^k \tag{2.9}$$

Definition 2.2. A connection Γ_{ij}^k is said to be *special* if its symmetric part Λ_{ij}^k coincides with the Christoffel symbol $\{{}^{\,k}_{i\,j}\}$ defined by h_{ij}: that is,

$$\Gamma_{ij}^k = \{{}^{\,k}_{i\,j}\} + S_{ij}{}^k \tag{2.10}$$

Definition 2.3. A connection Γ_{ij}^k is said to be a *semisymmetric k-connection*, or briefly *(k)-connection*, if its torsion tensor $S_{ij}{}^k$ is of the form

$$S_{ij}{}^k = 2\delta_{[i}^k X_{j]} + k_{ij} Y^k \tag{2.11}$$

for some vectors X_i and Y_i.

Definition 2.4. A connection Γ_{ij}^k is said to be *Einstein* if it satisfies the following Einstein equations:

$$\partial_k g_{ij} - g_{mj}\Gamma_{ik}^m - g_{im}\Gamma_{kj}^m = 0 \tag{2.12a}$$

or equivalently

$$D_k g_{ij} = 2S_{kj}{}^m g_{im} \tag{2.12b}$$

where D_k is the symbolic vector of the covariant derivative with respect to Γ_{ij}^k.

Hlavatý (1957) proved the following theorem:

Theorem 2.5. If equations (2.12) admit a solution Γ_{ij}^k, then this solution must be of the form

$$\Gamma_{ij}^k = \{{}^{\,k}_{i\,j}\} + U_{ij}^k + S_{ij}{}^k \tag{2.13}$$

where

$$U_{ij}^k = 2h^{ka} S_{a(i}{}^m k_{j)m} \tag{2.14}$$

In subsequent considerations we shall need the following scalars:

$$\alpha = G/H \tag{2.15a}$$

$$\beta = K/H \tag{2.15b}$$

3. $E(k)$-CONNECTION AND $SE(k)$-CONNECTION

In this section, we introduce the concepts of $E(k)$-connection and $SE(k)$-connection, and obtain the necessary and sufficient condition that there exists a unique $SE(k)$-connection in X_n.

Definition 3.1. In X_n, a connection Γ^k_{ij} is called an $E(k)$-*connection* if it is both Einstein and a (k)-connection.

Theorem 3.2. If there is an $E(k)$-connection Γ^k_{ij} in X_n, then it must be of the form

$$\Gamma^k_{ij} = \left\{ {k \atop i\,j} \right\} + 2k_{(i}{}^k X_{j)} - 2k_{(i}{}^k k_{j)m} Y^m + 2\delta^k_{[i} X_{j]} + k_{ij} Y^k \tag{3.1}$$

for some vectors X_i and Y_i.

Proof. Suppose that there is an $E(k)$-connection Γ^k_{ij} in X_n, then it is given by (2.13) and its torsion tensor $S_{ij}{}^k$ is given by (2.11) for some vectors X_i and Y_i. Substituting (2.11) into (2.14), we have

$$U^k_{ij} = 2k_{(i}{}^k X_{j)} - 2k_{(i}{}^k k_{j)m} Y^m \tag{3.2}$$

Substituting (2.11) and (3.2) into (2.13), we have (3.1).

Theorem 3.3. Suppose that in X_n, there is an $E(k)$-connection (3.1) for some vectors X_i and Y_i. Then the $E(k)$-connection is special if and only if the vectors X_i and Y_i are related by

$$X_i = k_{ij} Y^j \qquad \text{(or equivalently } Y_j = {}^*k_{ij} X^i\text{)} \tag{3.3}$$

Proof. In virtue of definition (2.2), the $E(k)$-connection (3.1) is special if and only if

$$2k_{(i}{}^k X_{j)} - 2k_{(i}{}^k k_{j)m} Y^m = 0 \tag{3.4}$$

If the vectors X_i and Y_i are related by (3.3), then we have (3.4) and so the $E(k)$-connection (3.1) is special. Conversely, suppose that the $E(k)$-connection is special; then we have (3.4). Contracting for i and k, we have

$$k_j{}^k (X_k - k_{km} Y^m) = 0$$

According to (2.3c), we have

$$X_k - k_{km} Y^m = 0$$

which implies (3.3).

Definition 3.4. A connection Γ^k_{ij} in X_n is called an *$SE(k)$-connection* if it is a special $E(k)$-connection.

Theorem 3.5. If there is an $SE(k)$-connection Γ^k_{ij} in X_n, then it must be of the form

$$\Gamma^k_{ij} = \{{}^{\,k}_{i\,j}\} + 2\delta^k_{[i}X_{j]} + k_{ij}Y^k \tag{3.5a}$$

for some vectors X_i and Y_i such that

$$X_i = k_{ij}Y^j \tag{3.5b}$$

Proof. Our assertion immediately follows from Theorems 3.2 and 3.3 and Definition 3.4.

Theorem 3.6. If there is an $SE(k)$-connection Γ^k_{ij} in X_2, then it coincides with the Christoffel symbol $\{{}^{\,k}_{i\,j}\}$.

Proof. Let Γ^k_{ij} be an $SE(k)$-connection in X_2; then it is of the form (3.5). Hence, its torsion tensor is given by

$$S_{ij}{}^k = 2\delta^k_{[i}k_{j]m}Y^m + k_{ij}Y^k$$

for some vector Y_j. Now we can easily check $S_{ij}{}^k = 0$; for instance,

$$S_{12}{}^1 = \delta^1_1 k_{2m}Y^m - \delta^1_2 k_{1m}Y^m + k_{12}Y^1 = 0$$

Consequently, we have $\Gamma^k_{ij} = \{{}^{\,k}_{i\,j}\}$.

Agreement 3.7. In our further considerations in the present paper, we restrict ourselves to the case $n \geq 4$, that is, $n = 4, 6, 8, \ldots$.

Lemma 3.8. There is an $SE(k)$-connection Γ^k_{ij} in X_n if and only if there is a vector X_i in X_n such that

$$\nabla_k k_{ij} = 2h_{k[i}X_{j]} - 2k_{k[i}Y_{j]} \tag{3.6}$$

where $X_i = k_{ij}Y^j$ and ∇_k is the symbolic vector of the covariant derivative with respect to $\{{}^{\,k}_{i\,j}\}$.

Proof. Suppose that Γ^k_{ij} is an $SE(k)$-connection in X_n; then it is of the form (3.5) and satisfies (2.12). Substituting (2.2) and (3.5) into (2.12a), we have (3.6). Conversely, suppose that in X_n there is a vector X_i satisfying (3.6). With this vector X_i, define a connection Γ^k_{ij} by (3.5). Then it is special and a (k)-connection. Since this connection satisfies (2.12a) in virtue of our assumption (3.6), it is Einstein. Therefore this connection is an $SE(k)$-connection.

Theorem 3.9. There is a unique $SE(k)$-connection (3.5) in X_n if and only if the following condition, called the $SE(k)$-condition, is satisfied:

$$\nabla_k k_{ij} = \frac{2}{2-n} (h_{k[i}\delta^p_{j]} + k_{k[i}\, {}^*k_{j]}{}^p)\nabla_q k_p{}^q \tag{3.7}$$

If this condition is satisfied, then

$$X_i = \frac{1}{2-n} \nabla_k k_i^k \tag{3.8}$$

Proof. Suppose that there is a unique $SE(k)$-connection (3.5) in X_n; then, in virtue of Lemma 3.8, there is a vector X_i such that (3.6). Multiplying by h^{jm} on both sides of (3.6) and contracting for k and m, we have

$$\nabla_k k_i{}^k = (2-n)X_i \tag{3.9}$$

equivalent to (3.8). Equation (3.7) results from the substitution of (3.8) into (3.6). Conversely, suppose that (3.7) is satisfied. Define two vectors X_i and Y_i by

$$X_i = \frac{1}{2-n} \nabla_k k_i^k \quad \text{and} \quad Y_j = {}^*k_{ij} X^i$$

Then the condition (3.7) implies the condition (3.6). Hence, in virtue of Lemma 3.8, we have an $SE(k)$-connection. Suppose that there exists another $SE(k)$-connection,

$$\bar{\Gamma}^k_{ij} = \{{}^{\,k}_{i\,j}\} + 2\delta^k_{[i}\bar{X}_{j]} + k_{ij}\bar{Y}^k, \qquad \bar{Y}_j = {}^*k_{ij}\bar{X}^i \tag{3.10a}$$

$$\bar{X}_i \neq X_i \tag{3.10b}$$

Applying the same method used to obtain (3.8), we have

$$\bar{X}_i = \frac{1}{2-n} \nabla_k k_i{}^k = X_i$$

which contradicts the assumption (3.10b). This proves the uniqueness of the $SE(k)$-connection in X_n.

Corollary 3.10. In the case mentioned in the previous theorem, we have

$$\Gamma^k_{ij} = \{{}^{\,k}_{i\,j}\} + \frac{1}{2-n} (2\delta^k_{[i}\delta^p_{j]} + k_{ij}\, {}^*k^{pk})\, \nabla_q k_p{}^q \tag{3.11}$$

Proof. Our assertion immediately follows from (3.5) and (3.8).

4. $SE(k)$-MANIFOLD X_n

This section is devoted to the study of the geometrical properties on the $SE(k)$-manifold defined by the $SE(k)$-connection Γ_{ij}^k.

Definition 4.1. An *$SE(k)$-manifold* X_n is a generalized even-dimensional Riemannian space X_n in which the $SE(k)$-condition (3.7) is satisfied.

Remark 4.2. In virtue of Theorem 3.9 and Definition 4.1, there always exists an $SE(k)$-connection Γ_{ij}^k of the following form in the $SE(k)$-manifold X_n:

$$\Gamma_{ij}^k = \{{}^{\,k}_{i\,j}\} + 2\delta_{[i}^k X_{j]} + k_{ij} Y^k \tag{4.1}$$

where

$$X_i = \frac{1}{2-n} \nabla_k k_i{}^k, \qquad Y_j = {}^*k_{ij} X^i \tag{4.2}$$

and the vectors X_i and Y_i satisfy

$$\nabla_k k_{ij} = 2h_{k[i} X_{j]} - 2k_{k[i} Y_{j]} \tag{4.3}$$

Theorem 4.3. In the $SE(k)$-manifold X_n the following relations hold:

$$X_i Y^i = 0 \tag{4.4a}$$

$$S_{ij}{}^k X_k = 0 \tag{4.4b}$$

$$S_{ij}{}^k Y^j = 0 \tag{4.4c}$$

$$S_{ij}{}^k k_k{}^j = 0 \tag{4.4d}$$

$$S_{ij}{}^k\, {}^*k_k{}^j = 0 \tag{4.4e}$$

where $S_{ij}{}^k$ is the torsion tensor of the $SE(k)$-connection Γ_{ij}^k.

Proof. Since $X_i Y^i = k_{ij} Y^j Y^i$ and k_{ij} is skew-symmetric, we have (4.4a). Making use of (2.11), (3.5b), and (4.4a), we have (4.4b)–(4.4e).

Theorem 4.4. In the $SE(k)$-manifold X_n, the torsion vector S_i $(= S_{ik}{}^k)$ has the following properties:

$$S_i = (2-n) X_i \tag{4.5}$$

$$D_j S_i = \nabla_j S_i \tag{4.6}$$

Proof. Putting $j = k$ in (2.11) and making use of (3.5b), we have (4.5). Making use of (4.4b) and (4.5), we have (4.6).

Theorem 4.5. In the $SE(k)$-manifold X_n, the torsion tensor $S_{ij}{}^k$ satisfies the following relation:

$$S_{ijk} = \tfrac{1}{2}(\nabla_i k_{kj} + \nabla_j k_{ik} + \nabla_k k_{ij}) \tag{4.7}$$

Proof. Making use of (4.3) and (2.11), we have

$$\nabla_i k_{kj} + \nabla_j k_{ik} + \nabla_k k_{ij} = 2(h_{ik} X_j - h_{jk} X_i + k_{ij} Y_k) = 2S_{ijk}$$

Theorem 4.6. In the $SE(k)$-manifold X_n, the scalars α and β, defined by (2.15a) and (2.15b), are constants.

Proof. Multiplying by $^*g^{ij}$, defined by (2.5), on both sides of (2.12a) and making use of (2.5) and (2.6), we have

$$\partial_k \ln G - \Gamma^m_{mk} - \Gamma^m_{km} = 0 \tag{4.8}$$

On the other hand, making use of (4.1) and the classical result $2\{{}_m{}^m{}_k\} = \partial_k \ln H$, we have

$$\Gamma^m_{mk} + \Gamma^m_{km} = 2\{{}_m{}^m{}_k\} = \partial_k \ln H \tag{4.9}$$

Substituting (4.9) into (4.8), we have

$$\partial_k \ln G - \partial_k \ln H = 0, \qquad \text{or} \qquad \partial_k \ln \alpha = 0$$

which proves that α is constant. Next, in virtue of (2.12b), we have

$$D_k k_{ij} = 2S_{k[j}{}^m g_{i]m} \tag{4.10}$$

Multiplying by $^*k^{ij}$, defined by (2.7), on both sides of (4.10) and making use of (2.2), (2.7), and (4.4e), we have

$$\partial_k \ln K - 2\Gamma^m_{mk} = 2S_k \tag{4.11a}$$

or equivalently

$$\partial_k \ln K - \Gamma^m_{mk} - \Gamma^m_{km} = 0 \tag{4.11b}$$

Substituting (4.9) into (4.11b), we have $\partial_k \ln \beta = 0$, which proves that β is constant.

Remark 4.7. Hlavatý (1957) also proved that the relation (4.4) and $\alpha =$ const on a manifold in which a special Einstein connection is connected.

5. FIELD EQUATIONS IN $SE(k)$-MANIFOLD X_n

By field equations we mean a set of partial differential equations for g_{ij}. This section is concerned with the geometry of the field equations in the $SE(k)$-manifold X_n.

Remark 5.1. Einstein's n-dimensional unified field theory is based on the following three principles as indicated by Hlavatý (1957):

Principle A. The algebraic structure is imposed on a generalized n-dimensional Riemannian space X_n by a general real tensor g_{ij} defined by (2.2).

Principle B. The differential geometrical structure on X_n is imposed by the tensor g_{ij} through the Einstein connection Γ_{ij}^k defined by a system of Einstein equations (2.12).

Principle C. In order to obtain g_{ij} involved in the solution for Γ_{ij}^k in (2.12), some conditions are imposed, which may be condensed to

$$S_i = 0 \tag{5.1a}$$

$$R_{[ij]} = \partial_{[i} Z_{j]} \tag{5.1b}$$

$$R_{(ij)} = 0 \tag{5.1c}$$

where Z_j is an arbitrary vector, and R_{ij} is the *contracted curvature tensor* defined by

$$R_{ij} = R^m{}_{ijm} \tag{5.2}$$

and $R^m{}_{ijk}$ is the *curvature tensor* of Γ_{ij}^k defined by

$$R^m{}_{ijk} = \partial_j \Gamma_{ik}^m - \partial_k \Gamma_{ij}^m + \Gamma_{ik}^p \Gamma_{pj}^m - \Gamma_{ij}^p \Gamma_{pk}^m \tag{5.3}$$

Remark 5.2. In virtue of Definition 4.1 and Remark 4.2, our $SE(k)$-manifold X_n is based on the principles A and B.

Theorem 5.3. The $SE(k)$-curvature tensor $R^m{}_{ijk}$ in the $SE(k)$-manifold X_n is given by

$$\begin{aligned} R^m{}_{ijk} = {} & H^m{}_{ijk} + \delta_i^m (\nabla_j X_k - \nabla_k X_j) + \delta_j^m (\nabla_k X_i + X_i X_k) \\ & - \delta_k^m (\nabla_j X_i + X_i X_j) + k_{ik} (\nabla_j Y^m - Y_j Y^m - X_j Y^m) \\ & - k_{ij} (\nabla_k Y^m - Y_k Y^m - X_k Y^m) + (h_{ij} X_k - h_{ik} X_j) Y^m \\ & + 2k_{jk} (Y_i Y^m + X_i Y^m) \end{aligned} \tag{5.4}$$

where $H^m{}_{ijk}$ is the curvature tensor defined by the Christoffel symbol $\{{}^{\,k}_{i\,j}\}$.

Proof. Substituting the relation (2.10) into (5.3), we have

$$R^m{}_{ijk} = H^m{}_{ijk} + \nabla_j S_{ik}{}^m - \nabla_k S_{ij}{}^m + S_{ik}{}^p S_{pj}{}^m - S_{ij}{}^p S_{pk}{}^m \tag{5.5}$$

Substituting (2.11) into (5.5) and making use of (4.2)–(4.4), we have (5.4).

Theorem 5.4. The contracted $SE(k)$-curvature tensor R_{ij} $(= R^m{}_{ijm})$ in the $SE(k)$-manifold X_n is given by

$$R_{ij} = H_{ij} + 2\nabla_{[j} X_{i]} + (2-n)(\nabla_j X_i + X_i X_j) - k_{ij} \nabla_k Y^k \tag{5.6}$$

where $H_{ij} = H^m{}_{ijm}$.

Proof. Putting $m = k$ in (5.4) and making use of (4.2), (4.3), and (4.4a), we have (5.6).

Therorem 5.5. The field equations (5.1b) and (5.1c) are equivalent to

$$(4-n)\,\partial_{[j}X_{i]}-k_{ij}\,\nabla_k Y^k=\partial_{[i}Z_{j]} \tag{5.7a}$$

$$H_{ij}+(2-n)(\nabla_{(j}X_{i)}+X_iX_j)=0 \tag{5.7b}$$

Proof. From (5.6), we have

$$R_{[ij]}=(4-n)\,\partial_{[j}X_{i]}-k_{ij}\,\nabla_k Y^k \tag{5.8a}$$

$$R_{(ij)}=H_{ij}+(2-n)(\nabla_{(j}X_{i)}+X_iX_j) \tag{5.8b}$$

Comparing (5.1b) and (5.1c) and (5.8a) and (5.8b), we have (5.7a) and (5.7b).

Theorem 5.6. The requirement (5.1a) reduces the $SE(k)$-manifold X_n to a Riemannian manifold with $H_{ij}=0$, and k_{ij} might be identified with the tensor of the electromagnetic field.

Proof. In virtue of (4.5), the field equation (5.1a) implies $X_i=0$ and so, in virtue of (4.1), $\Gamma^k_{ij}=\{{}^{\,k}_{i\,j}\}$. Hence $R_{ij}=H_{ij}$, so that (5.1b) is automatically satisfied by $Z_i=\partial_i Z$, and (5.7b) reduces to $H_{ij}=0$. Hence the $SE(k)$-manifold X_n is a Riemannian manifold with $H_{ij}=0$. Furthermore, from (3.7) and (3.8), we have

$$\partial_{[k}k_{ij]}=0 \qquad \text{and} \qquad \nabla_k k_i{}^k=0$$

Hence k_{ij} may be identified with tensor of the electromagnetic field.

Remark 5.7. Hlavatý (1957) also proved that if

$$S_{ij}{}^k=\tfrac{2}{3}S_{[i}\delta^k_{j]} \tag{5.9}$$

then the requirement (5.1b) reduces space-time X_4 to a Riemannian manifold X_4 with $H_{ij}=0$.

Remark 5.8. The requirement (5.1a) is too strong in the field theory in our $SE(k)$-manifold X_n. If we exclude the condition (5.1a), or if we replace (5.1a) by another condition, then the integrability conditions of (5.7a) are given by

$$\partial_{[m}(k_{ij]}\,\nabla_k Y^k)=0 \tag{5.10}$$

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